LET $G = GL(n, \mathbb{C})$

CONSIDER A REP'N $\pi : G \to GL(m, \mathbb{C})$

ANALYTIC

$$T = \left\{ \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \right\} \subseteq G$$

RESTRICTING TO T IT BREAKS UP INTO ONE-DIM'L REPS SINCE T IS ABELIAN

IF $\lambda \in \mathbb{Z}^n = \Lambda$. LET US CONSIDER THE REP'N $t = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \to \prod t_i^{\lambda_i}$ OF T.

SUCH A REP'N IS CALLED A __WEIGHT__. EVERY IRR ANALYTIC REP'N OF T IS A WEIGHT. Λ = WEIGHT LATTICE

$$\pi|_T = \bigoplus_{\mu \in \Lambda} d_\mu \cdot (\text{WEIGHT } \mu).$$

WE CAN ALSO UNDERSTAND THIS IN TERMS OF THE CHARACTER

$$\chi_\pi(g) = \mathrm{tr}\, \pi(g)$$

$$\chi_\pi(t) = \sum_{\mu \in \Lambda} d_\mu t^\mu$$

$$t^\mu := \prod t_i^{\mu_i}.$$

IF π IS IRREDUCIBLE $\chi_\pi(t)$ IS A SCHUR POLYNOMIAL.

THIS IS A SYMMETRIC POLYNOMIAL IN $t_1, \dots, t_n$

BECAUSE IF $\sigma \in S_n$ LET σ BE CONSIDERED TO BE A PERMUTATION MATRIX

$$\sigma = (1\,2\,3) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in GL(n, \mathbb{C})$$

$S_n \subseteq N_G(T)$ ACTS ON T BY

CONJUGATION. (WE WOULD HAVE TO BE A LITTLE MORE CAREFUL FOR OTHER GROUPS E.G. S_n)

$S_n \cong N_G(T)/T$ IT HAPPENS FOR $GL(n)$ THAT G CONTAINS A SUBGROUP ISOMORPHIC TO S_n.

$$\chi_\pi(hgh^{-1}) = \operatorname{tr} \pi(hgh^{-1})$$
$$= \operatorname{tr} \pi(g) = \chi_\pi(g).$$

IF $g = t \in T$, $h = \sigma \in S_n$

$$\chi_\pi(\sigma t \sigma^{-1}) = \chi_\pi(t) = \underset{\substack{\uparrow \\ \text{SOME POLY}}}{\chi}(t_1, \ldots, t_n)$$

$$\| $$

$$\chi_\pi(t_{\sigma^{-1}(1)}, \ldots, t_{\sigma^{-1}(n)})$$

THE GROUP $W = N_G(T)/T = S_n$ IS CALLED THE WEYL GROUP.

ON FRIDAY WE CONSIDERED OPERATORS

$$e_i : V \to V$$

$$f_i : V \to V$$

$$e_i(v) = \frac{d}{dt} \pi\left(e^{tE_{i,i+1}}\right) v \Big|_{t=0}$$

$E_{i,j}$ = MATRIX WITH 1 IN (i,j) POS
0 ELSEWHERE

$$f_i(v) = \frac{d}{dt} \pi\left(e^{tE_{i+1,i}}\right) v \Big|_{t=0}$$

IT IS A CALCULATION THAT WITH

$\alpha_1 = (1, -1, 0, \dots) \in \Lambda$

$\alpha_2 = (0, 1, -1, 0, \dots) \in \Lambda$

$\vdots$

$\alpha_{n-1} = (0, 0, \dots, 1, -1) \in \Lambda$

e_i, f_i SHIFT THE WEIGHT.

$$V = \bigoplus_{\text{WEIGHT } \mu} V_\mu$$

$$V_\mu = \{ v \in V \mid \pi(t) v = t^\mu \cdot v \}$$

$t \in T$.

$$\text{DIM}(V_\mu) = d_\mu .$$

$$e_i(V_\mu) \subseteq V_{\mu + \alpha_i} \quad \text{(MIGHT BE ZERO)}$$

$$f_i(V_\mu) \subseteq V_{\mu - \alpha_i}$$

$G = GL(3)$ 2 IRREDUCIBLE REPS OF DEGREE 3.

$$\pi_{\text{STANDARD}} : GL(3, \mathbb{C}) \xrightarrow[\text{IDENTITY MAP!}]{} GL(3, \mathbb{C})$$

$$\mu = (1,0,0) \quad \text{or} \quad (0,1,0) \quad \text{or} \quad (0,0,1)$$

$$V_\mu = \begin{pmatrix} * \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ * \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 0 \\ * \end{pmatrix}$$

$$e_{\cdot\nu} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{d}{dt}\left(\exp\begin{pmatrix} 0 & t & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right)\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{t=0}$$

$$= \frac{d}{dt}\begin{pmatrix} 1 & t & 0 \\ 0 & \cancel{0} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{t=0}$$

$$\frac{d}{dt}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\Bigg|_{t=0} = 0$$

$$e_{\cdot\nu}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{d}{dt}\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

THIS CALCULATION SHOWS

$$V_{(1,0,0)} = \mathbb{C}\begin{pmatrix}1\\0\\0\end{pmatrix} \xrightarrow{e_1} \text{zero}$$

$$V_{(0,1,0)} = \mathbb{C}\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$e\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

CALL THESE VECTORS

$$\begin{pmatrix}1\\0\\0\end{pmatrix} = \boxed{1}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} = \boxed{2}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} = \boxed{3}$$

$$e_1\boxed{1} = 0 \qquad e_1\boxed{2} = \boxed{1} \qquad e_1\boxed{3} = 0$$

$$e_2\boxed{1} = 0 \qquad e_2\boxed{2} = 0 \qquad e_2\boxed{3} = \boxed{2}$$

SIMILARLY

$$\left.\begin{aligned} f_1 \boxed{1} &= \boxed{2} \\ f_2 \boxed{2} &= \boxed{3} \end{aligned}\right\} \quad \text{ALL OTHER } f_i \boxed{j} = 0$$

DRAW THE "CRYSTAL GRAPH" WITH CONVENTION

$$\overset{x}{\bullet} \xrightarrow{\;i\;} \overset{y}{\bullet}$$

MEANS $f_i(x) = y$ OR $e_i(y) = x$

ANOTHER FIB: THESE CONDITIONS MAY NOT BE EQUIVALENT.

$$\boxed{1} \xrightarrow{\;1\;} \boxed{2} \xrightarrow{\;2\;} \boxed{3}$$

WE ARE SLIGHTLY MISREPRESENTING THE NATURE OF THE MAPS e_i, f_i

IT MAY NOT BE POSSIBLE IF V_μ HAS DIMENSION ≥ 2 TO FIND A BASIS THAT IS WELL-BEHAVED FOR ALL e_i, f_i.

ANOTHER EXAMPLE:

$$\Pi_{(1,1,0)} : GL(3) \longrightarrow GL(3)$$

$$\Pi_{(1,1,0)}(g) = \det(g) \cdot {}^t g^{-1}$$

EIGENVECTORS FOR $\begin{pmatrix} t_1 & & \\ & t_2 & \\ & & t_3 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow t_1 t_2 t_3 \begin{pmatrix} t_1^{-1} \\ 0 \\ 0 \end{pmatrix} = t_2 t_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

WEIGHT $= (0,1,1)$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad t_1 t_2 t_3 \begin{pmatrix} 0 \\ t_2^{-1} \\ 0 \end{pmatrix} = t_1 t_3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \rightsquigarrow \qquad t_1 t_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \boxed{\tfrac{2}{3}} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \boxed{\tfrac{1}{3}} \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\tfrac{1}{2}}$$

$$\boxed{\begin{matrix}1\\2\end{matrix}} \xrightarrow{2} \boxed{\begin{matrix}1\\3\end{matrix}} \xrightarrow{1} \boxed{\begin{matrix}2\\3\end{matrix}} \quad \checkmark$$

LET'S CHECK $f_2 \boxed{\begin{matrix}1\\3\end{matrix}} = \boxed{\begin{matrix}1\\2\end{matrix}}$ $\qquad f_i = \exp(tE_{i+1,i})$

$$x \xrightarrow{i} y \qquad f_i(x) = y$$

$$f_2\left(\boxed{\begin{matrix}1\\3\end{matrix}}\right) = \frac{d}{dt}\begin{pmatrix}1&&\\&1&\\&t&1\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\Bigg|_{t=0}$$

$$\frac{d}{dt}\begin{pmatrix}0\\1\\t\end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

REPS OF $SL(3)$

I. STANDARD $\qquad$ CHAR : $D_{(1,0,0)} =$

$t_1 + t_2 + t_3$

$$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \boxed{3} \qquad \text{DIM} = 3$$

II. EXTERIOR SQUARE   CHAR : $s_{(1,1,0)}$

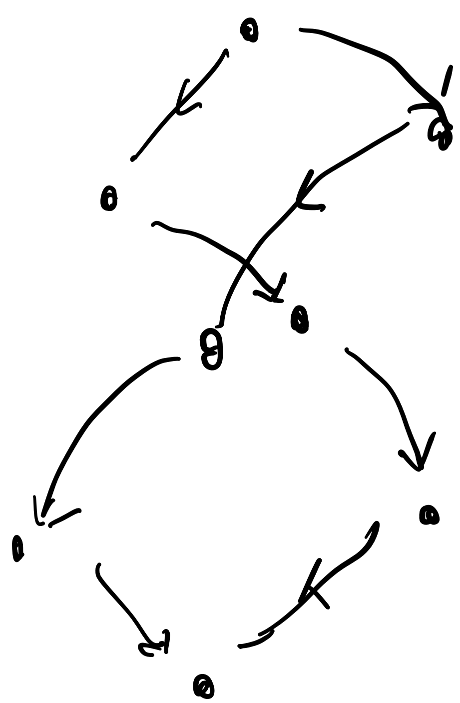
$= t_1 t_2 + t_2 t_3 + t_1 t_3$

$$\boxed{\begin{matrix}1\\2\end{matrix}} \xrightarrow{2} \boxed{\begin{matrix}1\\3\end{matrix}} \xrightarrow{1} \boxed{\begin{matrix}1\\3\end{matrix}}$$

DIM 3

SHAPE = $(1,1)$

III. ADJOINT DIM 8



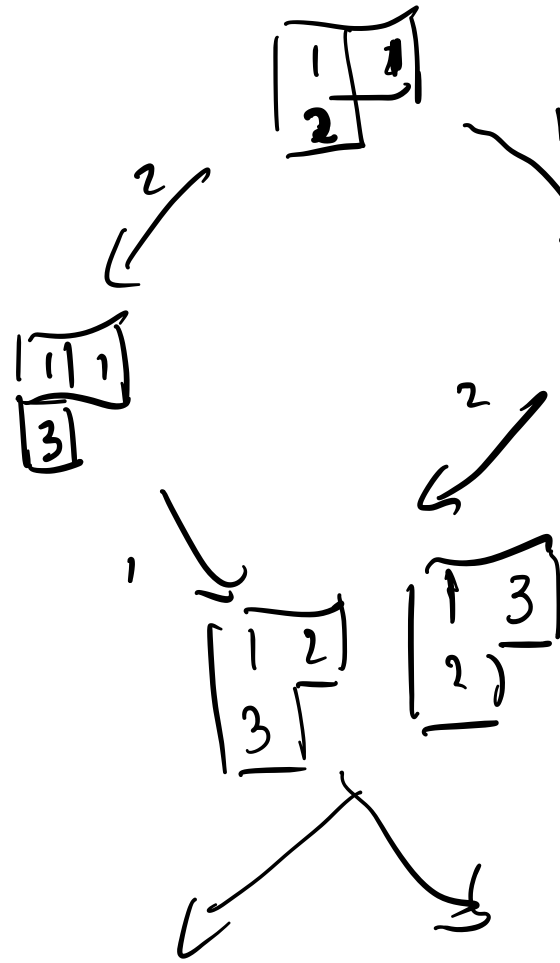
$\pi(g)$ ACTS ON

3×3 MATS OF

TRACE ZERO

BY CONJUGATION

LAST WEEK



THESE TWO HAVE SAME WEIGHT.

SSYT (SEMISTANDARD YOUNG TABLEAU)

IS A FILLING OF A SHAPE (PARTITION) BY INTEGERS

ROWS WEAKLY INCREASING COLUMNS STRICTLY INCREASING.

DEFINITION: A CRYSTAL IS A SET $\mathcal{C}$ WITH A MAP

$$wt: \mathcal{C} \to \Lambda \quad (\cong \mathbb{Z}^n)$$

OPERATORS $e_i, f_i : \mathcal{C} \to \mathcal{C} \cup \{0\}$

$e_i(x) = 0$ MEANS $e_i(x)$ IS NOT DEFINED AS AN ELEMENT OF $\mathcal{C}$.

$$wt(e_i(x)) = wt(x) + \alpha_i \text{ IF } e_i(x) \neq 0$$

$$wt(f_i(x)) = wt(x) - \alpha_i \text{ IF } f_i(x) \neq 0$$

$e_i(x) = y$ IS EQUIV. TO $f_i(y) = x$.

ROUGHLY TAKE A REP'N OF $GL(n, \mathbb{C})$ FIND A GOOD BASIS (MAY NOT BE POSSIBLE WHICH IS WHY WE SAY ROUGHLY).

$$e_i(x) = \frac{d}{dt}\left(\exp\left(tE_{i,i+1}\right)x\right)\Big|_{t=0}$$

$$f_i(x) \quad \text{''} \quad E_{i+1,i} \quad \text{''}$$

ALTHOUGH THIS DOES NOT WORK IT CAN BE FIXED THROUGH THE THEORY OF QUANTUM GROUPS.

IF C_1, C_2 ARE CRYSTALS $C_1 \otimes C_2$ CAN BE DEFINED COMBINATORIALLY

THEOREM. FOR EVERY "DOMINANT WEIGHT" λ

DOMINANT: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

IF $\lambda_n \geq 0$ THEN λ IS A PARTITION

THERE IS A CRYSTAL $\mathcal{B}_\lambda$

AND AN IRR REP'N π_λ WITH

$\Delta\lambda$ (SCHUR POLYNOMIAL)

$$\sum_{x \in \mathcal{B}_\lambda} t^{wt(x)} = \Delta_\lambda(t)$$

$$\pi_\lambda \otimes \pi_\mu = \sum c_{\lambda\mu}^{\nu} \pi_\nu$$

LITTLEWOOD RICH. COEFS

$$B_\lambda \otimes B_\mu = \sum c^{\nu}_{\lambda\mu} B_\lambda .$$

DISJOINT UNION.

EXTENDS TO ALL REDUCTIVE LIE GROUPS.

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{pmatrix} 0 & 0 & 0 \\ \bullet & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

3 x 3 MATRIX WITH TRACE ZERO

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}$$

GROUP ACTION IS CONJUGATION

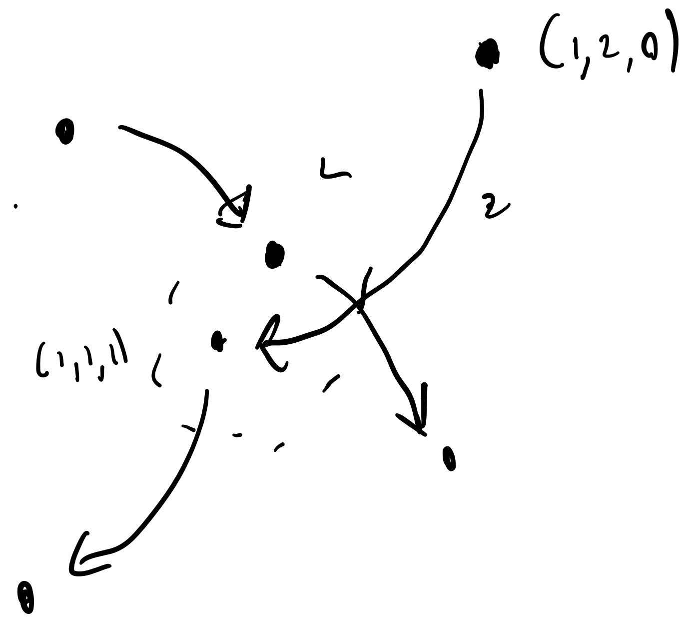
$$f_2 \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \checkmark$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & -t & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix}$$

$$\frac{d}{dt} \exp(tE_{3,2}) \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

WE WILL GET INTO TROUBLE WITH THE WEIGHT $(2,1,0)$

WEIGHT SPACE IS 2-DIM'L AND DOESN'T HAVE A GOOD BASIS.



$$f_2 \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array} = \begin{array}{cc} 1 & 3 \\ 2 & \end{array} \quad \text{or} \quad \begin{array}{cc} 1 & 2 \\ 3 & \end{array}$$